Fateev-Zamolodchikov and Kashiwara-Miwa models: boundary star-triangle relations and surface critical properties

Yu-Kui Zhou[1]

Department of Mathematics, School of Mathematical Sciences,
The Australian National University, Canberra ACT 0200, Australia

Abstract

The boundary Boltzmann weights are found by solving the boundary star-triangle relations for the Fateev-Zamolodchikov and Kashiwara-Miwa models. We calculate the surface free energies of the models. The critical surface exponent α_s of the Kashiwara-Miwa model is given and satisfies the scaling relation $\alpha_b = 2\alpha_s - 2$, where α_b is the bulk exponent.

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exactly solved models; boundary Boltzmann weights; boundary star-triangle relations; surface critical phenomena; surface free energies.

1 Introduction

Recently much research interest has been attracted to study exactly solvable models or integrable models with boundaries [1, 2, 3, 4, 5]. In statistical mechanics it has been shown that the non-periodic integrable lattice models with reflection boundaries are useful to exploit surface critical properties (for reviews see [6, 7] and references therein).

In the study of the six-vertex model [8] it has been made clear that the integrability of reflection boundary models is governed by the Yang-Baxter equation [9, 10] of the bulk Boltzmann weights and the boundary Yang-Baxter equation (or reflection equation [11]) of the boundary Boltzmann weights. The boundary Boltzmann weights have been calculated for many exactly solvable models [12]-[23]. The surface properties of these models deserve to be investigated ([24, 25]).

[1] on leave from Northwest University, China.

The Fateev-Zamolodchikov model [26], Kashiwara-Miwa model [27] and chiral Potts model [28, 29] are very interesting Z_N models in statistical mechanics. Of these models the bulk Boltzmann weights satisfy the star-triangle relations (see equ.(2.12)), the simplest form of the Yang-Baxter equation as stated in [30]. The integrable boundary weights and the corresponding surface behaviour of the Z_N-models have been not studied. In this paper the Fateev-Zamolodchikov and Kashiwara-Miwa models are considered. We construct the boundary Boltzmann weights and their boundary star-triangle relations, the simplest form of the boundary Yang-Baxter equation (or reflection equations). The corresponding surface free energies are calculated. The excess surface critical exponent of the Kashiwara-Miwa model is given. The chiral Potts model is not studied in this paper. This model will be considered separately because of the absence of spectral difference property.

The layout of this paper is as follows. In the next three sections a general consideration is treated. This means that the results given are applicable for the Fateev-Zamolodchikov model and Kashiwara-Miwa model, and for other Z_N models provided their bulk Boltzmann weights satisfy some required crossing and inversion relations. In section 2 we define the double-row transfer matrix with the left and right boundaries. Imposing the commutativity of the transfer matrix with different spectral parameters, the boundary star-triangle relations are extracted. The left and right boundary Boltzmann weights are determined by the boundary star-triangle relations. In section 3 we discuss the fusion procedure of the Z_N models briefly. Then the functional relations of the fused transfer matrices are given. In section 4 we show the spectral-independent[2] boundary Boltzmann weights. These boundary Boltzmann weights are used to construct a Z_N model with the fixed-spins along its boundaries. The star-triangle relation is enough to hold the integrability of the model with such boundary condition. In sections 5 and 6 we consider the Fateev-Zamolodchikov model and Kashiwara-Miwa model. Apart from the fixed-spin boundary conditions, we find another spectral-dependent boundary Boltzmann weights for the Fateev-Zamolodchikov model. In sections 7 the bulk and surface free energies of these two models are calculated. In particular, the surface critical exponent is extracted from the surface free energy deviated from criticality for the off-critical Kashiwara-Miwa model. Finally a brief discussion concludes the paper.

[2]spectral parameter independent

2 Integrability

The non-chiral Z_N model is generally represented by a planar lattice model with N-spins that live on the sites of the lattice and interact along edges. The Boltzmann weights are dependent on a spectral parameter u and a crossing parameter λ. Graphically they can be represented by

$$W(u|a,b) = \Big|_{b}^{a}\ {\scriptstyle W(u)} \qquad \overline{W}(u|a,b) = \Big|_{b}^{a}\ {\scriptstyle \overline{W}(u)}\ , \tag{2.1}$$

while the boundary Boltzmann weights are represented by

$$K_r(u;\xi_r|a,b) = \Big\}_{b}^{a}\ {\scriptstyle K(u;\,\xi_r)} \qquad K_l(u;\xi_l|a,b) = \Big\}_{b}^{a}\ {\scriptstyle K(\lambda\,-\,u;\,\xi_l)} \quad . \tag{2.2}$$

The spins $a, b = 1, 2, \cdots, N$. The left K_l and right K_r boundary weights may depend on an arbitrary parameter ξ_l and ξ_r respectively.

The Boltzmann weights share the no-chiral symmetry, or they are invariant under interchanging spins a and b in (2.1)-(2.2). We also suppose that they satisfy the crossing symmetry

$$W(u|a,b) = \overline{W}(\lambda - u|a,b) \tag{2.3}$$

and inversion relations

$$\sum_d \overline{W}(u|a,d)\overline{W}(-u|d,c) = {}_a\overset{\overline{W}(u)\quad\ \overline{W}(-u)}{\text{———}\bullet\text{———}}{}_c = \overline{\rho}(u)\delta_{a,c} \tag{2.4}$$

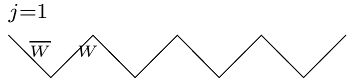
$$\sum_d W(\lambda-u|a,d)W(\lambda+u|d,c) = {}_a\overset{W(\lambda+u)\quad W(\lambda-u)}{\text{———}\bullet\text{———}}{}_c = \rho(u)\delta_{a,c} \tag{2.5}$$

$$W(-u|a,d)W(u|a,d) = {}_a\overset{W(u)\quad\ W(-u)}{\text{———}\underset{d}{\circ}\text{———}}{}_a = g(u)g(-u) \tag{2.6}$$

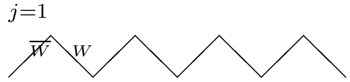
where the solid circles mean sum over all possible spins. Second inversion relation can be given from first one using the crossing symmetry. Thus $\rho(u) = \overline{\rho}(u)$.

Introduce two matrices with a single row of alternative Boltzmann weights W and $\overline{W}$ such as

$$\tilde{V}^{\sigma'_1\sigma'_{L+1}}(u)_{\psi,\phi'} = \prod_{j=1}^{L} \overline{W}(u|\sigma'_j,\sigma_j)W(u|\sigma_j,\sigma'_{j+1})$$

$$=$$  (2.7)

$$V^{\sigma_1\sigma_{L+1}}(u)_{\phi,\psi'} = \prod_{j=1}^{L} \overline{W}(u|\sigma_j,\sigma'_j)W(u|\sigma'_j,\sigma_{j+1})$$

$$=$$  (2.8)

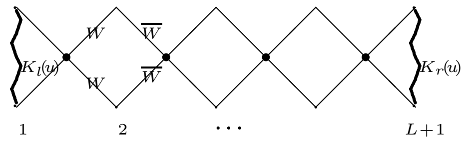


Figure 1: The transfer matrix $T(u)$: two rows of sites of the diagonal square lattice with the boundary weights $K_l(u;\xi_l)$ and $K_r(u;\xi_r)$, where the solid circles mean sum.

where spin sets $\phi = (\sigma_1, \sigma_2, \cdots, \sigma_L, \sigma_{L+1})$ and $\psi = (\sigma_1, \sigma_2, \cdots, \sigma_L)$. Then the transfer matrix $T(u)$ is defined by the following elements (see fig.1)

$$T(u)_{\phi,\phi'} \quad = \quad K_l(u|a,a')V^{ab}(u)\tilde{V}^{a'b'}(u)K_r(u|b,b'). \tag{2.9}$$

The transfer matrix is called as the double-row transfer matrix. Based on it the planar square lattice can be constructed. The boundaries on left and right sides of the lattice are not periodic. For such lattice the partition function is given by

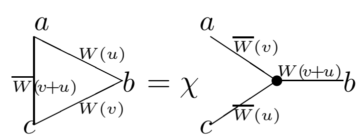
$$Z(u) = Tr\Big(T(u)\Big)^M, \tag{2.10}$$

where M is the number of the double-rows in the lattice.

The non-periodic boundary model is integrable if the transfer matrix forms commuting family

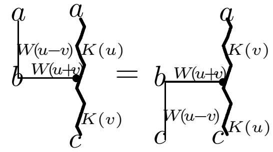
$$T(u)T(v) = T(v)T(u). \tag{2.11}$$

Suppose that the bulk Boltzmann weights satisfy the star-triangle relation



, (2.12)

where χ is a spin-independent parameter. The star-triangle relation is essential to guarantee the integrability of a model with a periodic boundary condition. For the boundary lattice model with the transfer matrix (2.9) the integrability requires both the star-triangle relation (2.12) for the bulk weights and the following boundary star-triangle relations

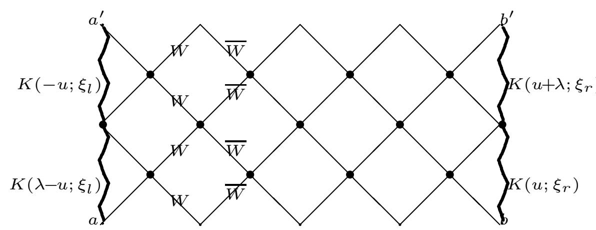


(2.13)

for the boundary weights. Given the bulk Boltzmann weights which solve (2.12), the boundary Boltzmann weights are determined by solving (2.13).

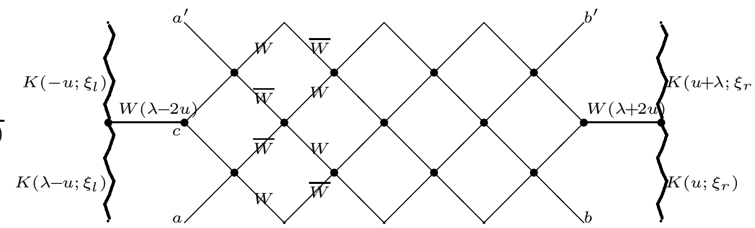
3 Functional relations and fusion hierarchies

Fusion procedure is the idea to build the new Boltzmann weights using the elementary Boltzmann weights [31]. The fused transfer matrices can be constructed with the new Boltzmann weights. Of particular, the fused transfer matrices form the fusion hierarchies and satisfy a group of functional relations. To apply the fusion to the Z_N model let us first consider the product of two transfer matrices $T(u)T(u+\lambda)$. Using a graph it can be represented by



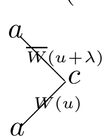
(3.1)

where the solid circles indicate sum over all possible spins. Inserting the inversion relation (2.5) into this and using the star-triangle relation, the product becomes

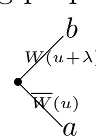
$\frac{1}{\rho(2u)}$ 

(3.2)

This can be divided into two terms according to the summation over c. If $c=a'$, using the inversion relations (2.4)-(2.6), namely the following properties

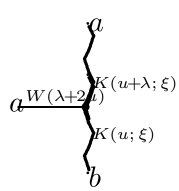


$= g(u)g(-u)$



$= \overline{\rho}(u)\delta_{a,b}$ (3.3)

for the bulk weights and



$= \rho_s^{(a)}(u;\xi)\delta_{a,b}$ (3.4)

for the boundaries, we are able to show this product gives the identity matrix multiplied by a function $f(u)$. The rest terms with $c \neq a'$ is collected together and is written as the fused transfer matrix $T^{(2)}(u)$. Thus we have

$$T(u)T(u+\lambda) = f(u) + T^{(2)}(u), \tag{3.5}$$

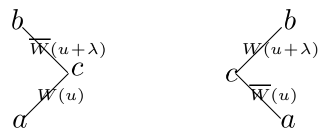
where

$$f(u) = (b(u))^2 s(u) \tag{3.6}$$

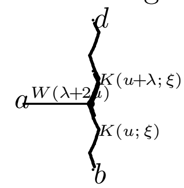
$$b(u) = [g(u)g(-u)\overline{\rho}(u)]^L \tag{3.7}$$

$$s(u) = \rho_s^{(a)}(u;\xi_r)\rho_s^{(a)}(-u;\xi_l). \tag{3.8}$$

The transfer matrix $T^{(2)}(u)$ is given by the following local fused weights



and



(3.9)

where $a \neq b$, $a \neq d$ and $c \neq N$. For clarity $T^{(2)}(u)$ can be represented by a similar graph in equation (3.2).

Consider n by m square lattice with the spectral parameters shifted properly [32]. We can fuse it to obtain the fused weights and thus define the fused transfer matrices $T^{(m,n)}(u)$, where two integers m, n greater than 1 are the fusion levels . For example, the above fused transfer matrix given in (3.5) corresponds to $T^{(1,2)}(u)$. Note that the fused boundary weights are labelled with a single fusion level n. Here we are not interested in the detail of showing the procedure of fusion. Instead, we write down the functional relations of the fused transfer matrices. Generalising (3.5), we are able to derive

$$T^{(m,n)}(u)T^{(m,1)}(u+n\lambda) = T^{(m,n+1)}(u) + f_{n-1}^{(m)} T^{(m,n-1)}(u), \tag{3.10}$$

where $T^{(m,0)}(u) = I$ and

$$f_n^{(m)} = f^{(m)}(u+n\lambda) \tag{3.11}$$

$$f^{(m)}(u) = s(u)b^{(m)}(u) \tag{3.12}$$

$$b^{(m)}(u) = \prod_{j=0}^{m-1} b(u-j\lambda)b(u+j\lambda) \tag{3.13}$$

The above functional relations imply

$$T^{(m,n)}(u)T^{(m,n)}(u+\lambda) = \prod_{k=0}^{n-1} f_k^{(m)} + T^{(m,n+1)}(u)T^{(m,n-1)}(u+\lambda). \tag{3.14}$$

The functional relations (3.10) and (3.14) are the T-system of the model. From them the y-system can be introduced. Define

$$y^{(m,0)}(u) = 0 \tag{3.15}$$

$$y^{(m,n)}(u) = T^{(m,n+1)}(u)T^{(m,n-1)}(u+\lambda) / \prod_{k=0}^{n-1} f_k^{(m)} , \tag{3.16}$$

From (3.14) we are able to derive the y-system

$$y^{(m,n)}(u)y^{(m,n)}(u+\lambda) = (1+y^{(m,n+1)}(u))(1+y^{(m,n-1)}(u+\lambda)) . \tag{3.17}$$

These functional relations are $su(2)$ type. They have been seen for other models with periodic boundaries [33, 34, 35, 32, 38] and reflection boundaries [36, 37, 19, 21, 6].

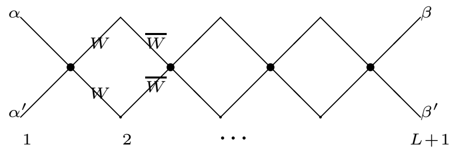


Figure 2: Two rows of sites of the square lattice with the boundary spins being fixed, which defines the transfer matrix $T_{(\alpha\alpha',\beta\beta')}(u)$.

4 Fixed-spin boundaries

A common future of the Z_N model is the existence of the following boundary weights

$$K(u|c,d) = \delta_{a,\alpha}\delta_{b,\beta} \tag{4.1}$$

for the given boundary spins α, β. Obviously, this K-matrix solves the boundary star-triangle relations (2.13). With this boundary weights the square lattice with fixed boundary spins can be constructed. The transfer matrix is given by Fig 2.

The functional relations given in the last section are still valid for the fixed boundary lattice. But the function $f(u)$ is changed. To see this let us consider the product of $T_{(\alpha\alpha',\beta\beta')}(u)T_{(\alpha'\alpha,\beta'\beta)}(u+\lambda)$ and this yields functional relation (3.5). The involved functions in the functional relations (3.5), (3.10) and (3.14)-(3.17) are given by (3.6) and (3.11) with

$$s(u) = W(\lambda - 2u|\alpha,\alpha')W(\lambda + 2u|\beta,\beta'), \tag{4.2}$$

while $b(u)$ and $b^{(m)}(u)$ remain unchanged.

5 Fateev-Zamolodchikov model

From this section we start to consider the specific non-chiral models. We apply the general formulae given previously to the Fateev-Zamolodchikov model in this section and to the Kashiwara-Miwa model in next section.

The Fateev and Zamolodchikov is described by the following bulk Boltzmann weights $W(u|a,b)$ and $\overline{W}(u|a,b)$ [26]

$$W(u|a,b) = g(u)\prod_{j=1}^{|a-b|}\frac{\sin((2j-1)\lambda - u)}{\sin((2j-1)\lambda + u)} \tag{5.1}$$

$$\overline{W}(u|a,b) = \overline{g}(u)\prod_{j=1}^{|a-b|}\frac{\sin(2(j-1)\lambda + u)}{\sin(2j\lambda - u)} \tag{5.2}$$

with

$$
\begin{aligned}
g(u) &= \prod_{j=1}^{n} \sin((2j-1)\lambda + u) && (5.3)\\
\overline{g}(u) &= \prod_{j=1}^{n} \sin(2j\lambda - u) && (5.4)
\end{aligned}
$$

where we use $n = (N-1)/2$ for odd N, $n = N/2$ for even N and $\lambda = \pi/2N$. For this model the parameter $\chi = \sqrt{N}$ in the star-triangle relation and $\overline{\rho}(u) = g(u)g(-u)\overline{g}(0)(g(0)\chi)^{-1}$ in the inversion relation (2.4).

The boundary Boltzmann weights are given by solving the boundary star-triangle relations (2.13). Apart from the fixed-spin boundary solution (4.1), we also present the following solution

$$
K(u;\xi|a,b) = K(u;\xi|a-b) = K(u;\xi|b-a) = 0, \qquad \text{unless } |a-b| = 0 \text{ or } N-1 \tag{5.5}
$$

and

$$
K(u;\xi|N-1) = CK(u;\xi|0)\sin(2u) \quad \text{and} \quad K(u;\xi|0) = \text{arbitrary} \tag{5.6}
$$

where C is u-independent and depends on ξ. The above solution implies that the function in (3.4)

$$
\rho_s^{(a)}(u;\xi) = g(\lambda + 2u)K(u|0)K(\lambda + u|0)(1 - C^2\epsilon\sin^2(2u)) \tag{5.7}
$$

and $s(u)$ is given by (3.8) for the Fateev-Zamolodchikov model, where $\epsilon = 1$ for $a = 1, N$ and $\epsilon = 0$ otherwise.

6 Kashiwara-Miwa model

The Kashiwara-Miwa model is the elliptic extension of the Fateev-Zamolodchikov model. The bulk Boltzmann weights are given by [27]

$$
\begin{aligned}
W(u|a,b) &= e^{-ug_{a,b}} g(u) \prod_{j=1}^{|a-b|} \frac{\vartheta_1((2j-1)\lambda - u)}{\vartheta_1((2j-1)\lambda + u)} \prod_{j=1}^{a+b+s} \frac{\vartheta_4((2j-1)\lambda - u)}{\vartheta_4((2j-1)\lambda + u)} && (6.1)\\
\overline{W}(u|a,b) &= e^{(u-\lambda)g_{a,b}} \overline{g}(u) \prod_{j=1}^{|a-b|} \frac{\vartheta_1(2(j-1)\lambda + u)}{\vartheta_1(2j\lambda - u)} \prod_{j=1}^{a+b+s} \frac{\vartheta_4(2(j-1)\lambda + u)}{\vartheta_4(2j\lambda - u)} && (6.2)
\end{aligned}
$$

with

$$
\begin{aligned}
g(u) &= \prod_{j=1}^{n} \vartheta_1((2j-1)\lambda + u)\vartheta_4((2j-1)\lambda + u) && (6.3)\\
\overline{g}(u) &= \prod_{j=1}^{n} \vartheta_1(2j\lambda - u)\vartheta_4(2j\lambda - u) && (6.4)
\end{aligned}
$$

where $g_{a,b} = \log(R_a R_b)/\lambda$ and $R_a = \sqrt{\vartheta_4(2s\lambda)/\vartheta_4(4a\lambda + 2s\lambda)}$. $s = 0$ or 1 for even N and $s = 0$ for odd N. The elliptic functions $\vartheta_1(u)$, $\vartheta_4(u)$ are standard theta functions of nome p

$$\vartheta_1(u) = \vartheta_1(u,p) = 2p^{1/4}\sin u \prod_{k=1}^{\infty}\left(1 - 2p^{2k}\cos 2u + p^{4k}\right)\left(1 - p^{2k}\right) \tag{6.5}$$

$$\vartheta_4(u) = \vartheta_4(u,p) = \prod_{k=1}^{\infty}\left(1 - 2p^{2k-1}\cos 2u + p^{4k-2}\right)\left(1 - p^{2k}\right) \tag{6.6}$$

where $0 < p < 1$ with $p = 0$ at criticality. The parameter $\chi = \overline{W}(0|0,0)/\overline{W}(\lambda|0,0)$ in the star-triangle relation and $\overline{\rho}(u) = g(u)g(-u)\overline{W}(\lambda|0,0)/W(0|0,0)$ in the inversion relation (2.4).

The boundary Boltzmann weights are given by (4.1). Unfortunately, we have not found other solutions for the Kashiwara-Miwa model. For the fixed-spin boundary the function $s(u)$ given in (4.2) is still valid.

7 Surface free energies

The surface free energies of the Z_N models together with their bulk free energies can be found from the functional relation (3.5) as the other models shown in [6, 37]. In $L \to \infty$ the fused transfer matrix $T^{(2)}(u)$ stands for the finite-size corrections of the transfer matrix $T(u)$. Therefore the bulk and surface free energies satisfy

$$T(u)T(u+\lambda) = f(u)\ , \tag{7.1}$$

The unitarity relation (7.1) combines the inversion relation and crossing symmetries of the local bulk and boundary face weights. We can separate the bulk free energy from the surface free energy [37, 38]. Let $T(u) = T_b(u)T_s(u)$ be the eigenvalues of the transfer matrix $\boldsymbol{T}(u)$. Define $T_b = \kappa_b^{2L}$ and $T_s = \kappa_s$, then the bulk and surface free energies are defined by $f_b(u) = -\log\kappa_b(u)$ and $f_s(u) = -\log\kappa_s(u)$ respectively. We have

$$\log T(u) = -2Lf_b(u) - f_s(u) + \ldots \quad \text{as} \quad L \to \infty. \tag{7.2}$$

The terms are represented by the dots are the finite-size corresctions.

In present paper we calculate the free energies $f_b(u)$ and $f_s(u)$ for the Fateev-Zamolodchikov model and Kashiwara-Miwa model. Seperating the surface free energy from the bulk one in (7.1), we are able to obtain

$$\kappa_b(u)\kappa_b(u+\lambda) = g(u)g(-u)\overline{\rho}(u) \tag{7.3}$$

$$\kappa_s(u)\kappa_s(u+\lambda) = s(u). \tag{7.4}$$

These inversion relations can be solved with certain analyticity assumptions. This is known as "inversion relation trick". The trick has been successfully applied to give the correct bulk free energy [39] and surface free energy [40] of the eight-vertex model.

7.1 Fateev-Zamolodchikov model

The bulk free energy of the Fateev-Zamolodchikov model has been given using the inversion relation trick in [26] and the root density method in [41]. Here the normalisation of the bulk Boltzmann weights (5.2) is different from that adopted in [26, 41], or $W(u|a,a) \neq 1$ and $\overline{W}(u|a,a) \neq 1$. So it is worthwhile exercise to derive the bulk free energy before calculating the surface free energy.

7.1.1 Bulk free energy

The bulk inversion relation (7.3) becomes

$$\kappa_b(u)\kappa_b(u+\lambda) = \prod_{j=1}^{n} \frac{\sin((2j-1)\lambda+u)\sin((2j-1)\lambda-u)}{\sin^2((2j-1)\lambda)} \tag{7.5}$$

for the Fateev-Zamolodchikov model. To solve it let us suppose that $\kappa(u)$ is analytic and non-zero in the regime $0<u<\lambda$. Changing variable $u=ix$ and applying the following Fourier transforms

$$\begin{aligned} K(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ [\ln\kappa(x)]''\ e^{ikx}, \\ [\ln\kappa(x)]'' &= \int_{-\infty}^{\infty} dk\ K(k)\ e^{-ikx} \end{aligned} \tag{7.6}$$

to (7.5), then solving the equation for $K(k)$ and inverting Fourier transform back to $\kappa(x)$, we have

$$[\ln\kappa(x)]'' = \int_{-\infty}^{\infty} dk\ k\ e^{(-ix+\lambda/2)k} \sum_{j=1}^{n} \frac{\cosh[((2j-1)\lambda-\pi/2)k]}{\sinh(k\pi/2)\cosh(k\lambda/2)}. \tag{7.7}$$

Integrating twice and using the condition $\kappa(u)=1$ as $u=0$ and $u=\lambda$, we obtain the bulk free energy

$$f_b(u) = -2\int_{-\infty}^{\infty} dk F_b(k,u), \tag{7.8}$$

where

$$F_b(k,u) = \frac{\sinh(ku)\sinh(k\lambda-ku)}{k\sinh(k\pi)\cosh(k\lambda)} \frac{\sinh(2kn\lambda)\cosh(2kn\lambda-k\pi)}{\sinh(2k\lambda)}. \tag{7.9}$$

7.1.2 Surface free energies

The surface free energies are consist of the excess and local surface free energies [43, 44, 45], or $f_s(u) = f_s^s(u) + f_s^l(u)$. Accordingly, the inversion relation (7.4) for the Fateev-Zamolodchikov model breaks into two parts determining the excess and local surface free energies respectively. Suppose

$$\kappa_s(u) = \kappa_s^s(u)\kappa_s^l(u).$$

Then the excess free energy reads

$$f_s^s(u) = -\log\kappa_s^s(u).$$

and the local free energy reads

$$f_s^l(u) = -\log\kappa_s^l(u).$$

Their inversion relations are given by

$$\begin{aligned} \kappa_s^s(u)\kappa_s^s(u+\lambda) &= \prod_{j=1}^{n}\frac{\sin(2j\lambda+u)\sin(2j\lambda-u)}{\sin^2(2j\lambda)} && (7.10)\\ \kappa_s^l(u)\kappa_s^l(u+\lambda) &= K(u|0)K(\lambda+u|0)K(-u|0)K(\lambda-u|0) \\ &\quad\times(1-C_l^2\sin^2(2u))(1-C_r^2\sin^2(2u)), && (7.11)\end{aligned}$$

where the local free energy occurs only for $a = 1, N$ in (5.7), or the boundary couplings appear. Furthermore, we can divide the local free energy into the left and right free energies,

$$f_s^l(u) = f_s^{ll}(u) + f_s^{lr}(u) \qquad (7.12)$$

$$\kappa_s^l(u) = \kappa_s^{ll}(u)\kappa_s^{lr}(u). \qquad (7.13)$$

Taking

$$C_l = 1/\sin\xi^l \quad C_r = 1/\sin\xi^r \qquad (7.14)$$

and the proper normalisation $K(u|0)$, we are able to obtain

$$\begin{aligned} \kappa_s^{ll}(u)\kappa_s^{ll}(u+\lambda) &= \frac{\sin(\xi^l-2u)\sin(\xi^l+2u)}{\sin^2\xi^l} && (7.15)\\ \kappa_s^{lr}(u)\kappa_s^{lr}(u+\lambda) &= \frac{\sin(\xi^r-2u)\sin(\xi^r+2u)}{\sin^2\xi^r}. && (7.16)\end{aligned}$$

As the bulk free energy, we suppose that $\kappa_s^s(u)$, $\kappa_s^{ll}(u)$ and $\kappa_s^{lr}(u)$ are analytic and non-zero in the regime $0 < u < \lambda$. Using the Fourier transforms (7.6), the inversion relations (7.10) and (7.15)-(7.16) can be solved similarly. Define two functions

$$F_s(k,u) = \frac{\sinh(ku)\sinh(k\lambda-ku)}{k\sinh(k\pi/2)\cosh(k\lambda)}\frac{\sinh(kn\lambda)\cosh(k\lambda(n+1)-k\pi/2)}{\sinh(k\lambda)} \qquad (7.17)$$

$$F_l(k,u,\xi) = \frac{\sinh(ku)\sinh(k\lambda-ku)\cosh(k\xi-k\pi/2)}{k\cosh(k\lambda)\sinh(k\pi/2)}. \qquad (7.18)$$

We are able to obtain the excess surface free energy

$$f_s(u) = -2\int_{-\infty}^{\infty} dk F_s(k,u), \tag{7.19}$$

the left local surface free energy

$$f_s^{ll}(u) = -2\int_{-\infty}^{\infty} dk F_l(k,u,\xi^l) \tag{7.20}$$

and the right local surface free energy

$$f_s^{lr}(u) = -2\int_{-\infty}^{\infty} dk F_l(k,u,\xi^r). \tag{7.21}$$

To derive the local free energies the surface coupling parameters have been restricted as $0 < \xi^{l,r} < \pi$.

7.2 Kashiwara-Miwa model

The function $s(u)$ and $b(u)$ are given by (4.2) and (3.7) respectively for the Kashiwara-Miwa model with fixed-spin boundaries. For simplicity, we take the boundary spins $\alpha' = \alpha$ and $s = 0$ in the bulk Boltzmann weights (6.1)-(6.2). We can write the inversion relations (7.3)-(7.4) as

$$\kappa_b(u)\kappa_b(u+\lambda) = \prod_{j=1}^{n} \frac{\vartheta_1((2j-1)\lambda+u)\vartheta_4((2j-1)\lambda+u)\vartheta_1((2j-1)\lambda-u)\vartheta_4((2j-1)\lambda-u)}{\vartheta_1((2j-1)\lambda)\vartheta_4((2j-1)\lambda)\vartheta_1((2j-1)\lambda)\vartheta_4((2j-1)\lambda)} \tag{7.22}$$

$$\kappa_s(u)\kappa_s(u+\lambda) = \prod_{j=1}^{n} \frac{\vartheta_1(2j\lambda+2u)\vartheta_4(2j\lambda+2u)\vartheta_1(2j\lambda-2u)\vartheta_4(2j\lambda-2u)}{\vartheta_1(2j\lambda)\vartheta_4(2j\lambda)\vartheta_1((2j-1)\lambda)\vartheta_4(2j\lambda)} \tag{7.23}$$

The bulk free energy has been calculated in [42]. Here we have taken the different normalisation. So we present the bulk free energy again along with the surface free energy.

To solve the unitarity relations (7.22)-(7.23) let us introduce the new variables

$$x = e^{-\pi\lambda/\epsilon}, \quad w = e^{-2\pi u/\epsilon}, \quad q = e^{-\pi^2/\epsilon} \quad p = e^{-\epsilon} \tag{7.24}$$

along with the conjugate modulus transformation of the theta functions,

$$\begin{aligned}\vartheta_1(u, e^{-\epsilon}) &= \rho(u,\epsilon)E\left(e^{-2\pi u/\epsilon}, e^{-2\pi^2/\epsilon}\right) & (7.25)\\ \vartheta_4(u, e^{-\epsilon}) &= \rho(u,\epsilon)E\left(-e^{-2\pi u/\epsilon}, e^{-2\pi^2/\epsilon}\right). & (7.26)\end{aligned}$$

The factor $\rho(u,\epsilon)$ is harmless and will be disregarded, while

$$E(z,x) = \prod_{n=1}^{\infty}(1-x^{n-1}z)(1-x^n z^{-1})(1-x^n). \tag{7.27}$$

Suppose that $\kappa_b(w)$ is analytic and nonzero in the annulus $x \le w \le 1$ and perform the Laurent expansion $\log \kappa_b(w) = \sum_{m=-\infty}^{\infty} c_m w^m$. Then inserting this into the logarithm of both sides of (7.22) and equating coefficients of powers of w gives

$$\begin{aligned} f_b(w,p) &= -\frac{2\pi}{\epsilon}\sum_{m=1}^{\infty}(1+(-1)^m)F_b(\pi m/\epsilon, u) \\ &= -\frac{4\pi}{\epsilon}\sum_{m=1}^{\infty} F_b(2\pi m/\epsilon, u) \end{aligned} \tag{7.28}$$

Similarly, Laurent expanding $\log \kappa_s(w) = \sum_{m=-\infty}^{\infty} c_m w^m$ and solving the crossing unitarity relation (7.23) under the same analyticity assumptions as for the bulk case gives the excess surface free energy

$$\begin{aligned} f_s(w,p) &= -\frac{4\pi}{\epsilon}\sum_{m=1}^{\infty}(1+(-1)^m)F_s(2\pi m/\epsilon, u) \\ &= -\frac{8\pi}{\epsilon}\sum_{m=1}^{\infty} F_s(4\pi m/\epsilon, u). \end{aligned} \tag{7.29}$$

The local surface free energy is not obtainable from the fixed-spin boundaries. Moreover, the excess surface free energy will be changed if the boundary spin $\alpha' \neq \alpha$.

7.3 Critical properties

The Kashiwara-Miwa model is more interesting here as their solution in terms of elliptic functions correspond to off-critical extensions of the the Fateev-Zamolodchikov model. The elliptic nome p measures the deviation from the critical point $p = 0$.

The critical behaviour of free energies is described by their the singular behaviour in $p \to 0$. In practice, the singular part of the free energies given in (7.28) and (7.29) are extracted by means of the Poisson summation formula [10]. For the bulk free energy $f_b(w,p)$ it follows that [42]

$$f_b(w,p) \sim p^N \log p \quad \text{as} \quad p \to 0. \tag{7.30}$$

For the surface free energy a similar treatment yields that

$$f_s(w,p) \sim \begin{cases} p^{N/2}\log p & N \text{ even} \\ p^{N/2} & N \text{ odd} \end{cases} \quad \text{as} \quad p \to 0. \tag{7.31}$$

According to the well developed phenomenology of critical behaviour at a surface [44, 45, 43], the excess surface critical exponent α_s can be obtained from the surface free

energy. Together with the bulk critical exponent α_b, which has already been given in [42], we have that

$$f_b(w,p) \sim p^{2-\alpha_b} \log p \quad \text{and} \quad f_s(w,p) \sim p^{2-\alpha_s} (\log p). \tag{7.32}$$

Comparing (7.32) with (7.30) and (7.31), we are able to obtain that

$$\alpha_b = 2 - N \tag{7.33}$$

$$\alpha_s = 2 - N/2 \tag{7.34}$$

and this verifies the scaling relation

$$\alpha_b = 2\alpha_s - 2. \tag{7.35}$$

The scaling relation (7.35) is consistent with the known scaling relation $\alpha_s = \alpha_b + \nu$ and $\alpha_b/2 = 1 - \nu$. Using the known scaling relation $\alpha_s = \alpha_b + \nu$ and $\alpha_1 = \alpha_b - 1$, the results (7.34) imply that the correlation length critical exponent $\nu = N/2$ and the local specific heat critical exponent $\alpha_1 = 1 - N$.

8 Discussion

In this paper the boundary star-triangle relations and the boundary weights for the Fateev-Zamolodchikov and Kashiwara-Miwa models have been studied. Of particular, it has been shown that the fixed-spin boundary lattices are always integrable provided the bulk Boltzmann weights satisfy the star-triangle relation. This is even true for the chiral Potts model.

The free energies of the Fateev-Zamolodchikov and Kashiwara-Miwa models have been calculated. The inversion relation determining the surface free energies is separated from the bulk one [37]. This leads to that Baxter's inversion trick [39] is applicable to calculating the surface free energy of integrable models with reflection boundaries. Of course, the calculation is correct because the analytic assumption is valid to the models. The surface critical exponent of the excess specific heat of the Kashiwara-Miwa model has been given. Using the relevant known scaling relations, the critical exponents of the correlation length ν and the local specific heat α_1 are predicted for the Kashiwara-Miwa model.

The study presented in this paper is to be extended to the chiral Potts model. The boundary Boltzmann weights of the model have been calculated and the further investigation is in progress.

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